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AUTHOR(S):

Hishida, Toshiaki

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A LOCAL EXISTENCE THEOREM FOR THE NAVIER-STOKES FLOW IN THE EXTERIOR TO A ROTATING OBSTACLE

TOSHIAKI HISHIDA 菱田 俊 明

Department of Applied Mathematics
Faculty of Engineering
Niigata University
Niigata 950-2102 Japan
(e-mail: hishida@eng.niigata-u.ac.jp)

ABSTRACT. Let us consider the three dimensional Navier-Stokes initial value problem in the exterior to a rotating obstacle. It is proved that a unique solution exists locally in time when the initial data a possess some regularity in the space L^2 (similarly to the assumption given by Fujita and Kato [4]) and satisfy $(\omega \times x) \cdot \nabla a \in H^{-1}$, where ω stands for the angular velocity of the rotating obstacle. An essential step for the proof is to deduce a certain smoothing property together with estimates near $t = 0$ of the semigroup (it is not an analytic one) generated by the operator $\mathcal{L}u = -P[\Delta u + (\omega \times x) \cdot \nabla u - \omega \times u]$, where P denotes the projection associated with the Helmholtz decomposition.

It is one of important problems in fluid mechanics to study the Navier-Stokes flow past a rotating obstacle. In order to understand the rotation effect mathematically, we will limit ourselves to a problem under the following simple situation; the angular velocity is constant and the translation is absent. In this article we discuss the locally in time existence of a unique solution to such a problem.

Let $\mathcal{O} \subset \mathbb{R}^3$ be a compact, isolated rigid obstacle which is bounded by a smooth surface Γ , and $\Omega = \mathbb{R}^3 \setminus \mathcal{O}$ the exterior domain occupied by a viscous incompressible fluid. Assume that the obstacle \mathcal{O} is rotating about the x_3 -axis with angular velocity $\omega = (0, 0, 1)^T$. Here and hereafter, super- T denotes the transpose and all vectors are column ones; $x = (x_1, x_2, x_3)^T$, $\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)^T$ and so on. Set

$$\Omega(t) = \{y = O(t)x; x \in \Omega\}, \quad \Gamma(t) = \{y = O(t)x; x \in \Gamma\},$$

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which actually vary as time t goes on (this is the situation under consideration) unless \mathcal{O} is axisymmetric, where

$$O(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We now consider the fluid motion around \mathcal{O} , which is governed by the initial boundary value problem for the Navier-Stokes equation

$$(NS.1) \quad \left\{ \begin{array}{ll} \partial_t w + w \cdot \nabla_y w = \Delta_y w - \nabla_y q, & y \in \Omega(t), t > 0, \\ \nabla_y \cdot w = 0, & y \in \Omega(t), t \geq 0, \\ w = \omega \times y, & y \in \Gamma(t), t > 0, \\ w \rightarrow 0, & |y| \rightarrow \infty, t > 0, \\ w(y, 0) = a(y), & y \in \Omega, \end{array} \right.$$

where $w = (w_1(y, t), w_2(y, t), w_3(y, t))$ and $q = q(y, t)$ denote, respectively, unknown velocity and pressure of the fluid. The boundary condition on $\Gamma(t)$ is the non-slip one since $dy/dt = \dot{O}(t)O(t)^T y = \omega \times y$, where $\dot{O}(t) = (d/dt)O(t)$. It is natural to reduce (NS.1) to the problem in the fixed domain Ω by using the coordinate system $x = O(t)^T y$ attached to the rotating obstacle. There are two ways to make the change of the fluid velocity. The one is

$$u(x, t) = O(t)^T w(y, t),$$

and the other is

$$v(x, t) = O(t)^T [w(y, t) - \omega \times y] = u(x, t) - \omega \times x.$$

We also make the change of the pressure by

$$p(x, t) = q(y, t).$$

Then we have

$$\begin{aligned}
\partial_t w &= O(t) \left[\partial_t u + \left(\dot{O}(t)^T O(t) x \right) \cdot \nabla_x u + O(t)^T \dot{O}(t) u \right] \\
&= O(t) [\partial_t u - (\omega \times x) \cdot \nabla_x u + \omega \times u] \\
&= O(t) [\partial_t v - (\omega \times x) \cdot \nabla_x v + \omega \times v], \\
\Delta_y w &= O(t) \Delta_x u = O(t) \Delta_x v, \\
\nabla_y q &= O(t) \nabla_x p, \\
\nabla_y \cdot w &= \nabla_x \cdot u = \nabla_x \cdot v,
\end{aligned}$$

and

$$\begin{aligned}
w \cdot \nabla_y w &= O(t) [u \cdot \nabla_x u] \\
&= O(t) [v \cdot \nabla_x v + (\omega \times x) \cdot \nabla_x v + \omega \times v + \omega \times (\omega \times x)].
\end{aligned}$$

The problem (NS.1) is thus reduced to the following (NS.2) and (NS.3) for $\{v, p\}$ and $\{u, p\}$, respectively. The former is the problem with not only the Coriolis force $2 \omega \times v$ but also the growing boundary condition at space infinity:

$$(\text{NS.2}) \left\{ \begin{array}{ll} \partial_t v + v \cdot \nabla_x v = \Delta_x v - 2 \omega \times v - \omega \times (\omega \times x) - \nabla_x p, & x \in \Omega, t > 0, \\ \nabla_x \cdot v = 0, & x \in \Omega, t \geq 0, \\ v = 0, & x \in \Gamma, t > 0, \\ v + \omega \times x \rightarrow 0, & |x| \rightarrow \infty, t > 0, \\ v(x, 0) = a(x) - \omega \times x, & x \in \Omega. \end{array} \right.$$

The latter is the problem with the convection term having the coefficient $\omega \times x$ which is understood as the rigid motion rotating about the x_3 -axis:

$$(NS.3) \quad \left\{ \begin{array}{ll} \partial_t u + u \cdot \nabla_x u = \Delta_x u + (\omega \times x) \cdot \nabla_x u - \omega \times u - \nabla_x p, & x \in \Omega, t > 0, \\ \nabla_x \cdot u = 0, & x \in \Omega, t \geq 0, \\ u = \omega \times x, & x \in \Gamma, t > 0, \\ u \rightarrow 0, & |x| \rightarrow \infty, t > 0, \\ u(x, 0) = a(x), & x \in \Omega. \end{array} \right.$$

Up to now the mathematical theory for the existence and uniqueness of solutions to the problem (NS.1) has been little developed. In his Habilitationsschrift [2] Borchers first attacked this problem, including the case where the angular velocity depends on time t . He dealt with the problem (NS.2) and proved the existence of weak solutions of class

$$v + \omega \times x (= u) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad \forall T > 0,$$

with the energy inequality provided that $a \in L^2(\Omega)$ satisfies

$$(1) \quad \nabla \cdot a = 0 \quad \text{in } \Omega, \quad \nu \cdot (a - \omega \times x) = 0 \quad \text{on } \Gamma,$$

where ν is the unit exterior normal vector to Γ . We donot know the uniqueness of weak solutions and this feature is the same as the standard Navier-Stokes theory. Later on, in [3] Chen and Miyakawa have treated (NS.3) for $\Omega = \mathbb{R}^3$, that is, the Cauchy problem. They have discussed the existence of weak solutions with the so-called strong energy inequality and some decay properties of the constructed solutions.

The purpose of the present article is to prove that there exists a unique local solution to the problem (NS.3) whenever the initial data $a \in L^2(\Omega)$ satisfying (1) possess some regularity and fulfill $(\omega \times x) \cdot \nabla a \in H^{-1}(\Omega)$.

To state our results precisely, we introduce notation. We use the same symbols for denoting the spaces of scalar and vector functions if there is no confusion. By $C_0^\infty(\Omega)$ we denote the class of all C^∞ functions with compact supports in Ω . Let $H^s(\Omega)$ for $s \geq 0$ be the usual L^2 Sobolev spaces. If s is not an integer, then the space $H^s(\Omega)$ is defined via the complex interpolation (see Lions and Magenes [11, Chapter 1]), that is,

$$H^s(\Omega) = [L^2(\Omega), H^m(\Omega)]_\theta, \quad s = \theta m, \quad m > 0 \text{ (integer)}, \quad 0 < \theta < 1.$$

The scalar product and the norm of $L^2(\Omega) = H^0(\Omega)$ are respectively denoted by (\cdot, \cdot) and $\|\cdot\|$. The space $H_0^s(\Omega)$, $s > 0$, is the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, and $H^{-s}(\Omega)$ stands for the dual space of $H_0^s(\Omega)$. Let $C_{0,\sigma}^\infty(\Omega)$ be the class of all solenoidal (that is, divergence free) vector functions whose components are in $C_0^\infty(\Omega)$. By $L_\sigma^2(\Omega)$ we denote the completion of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)$. Then the space $L^2(\Omega)$ of vector functions admits the following orthogonal decomposition, the Helmholtz decomposition (Temam [13, Chapter I]):

$$L^2(\Omega) = L_\sigma^2(\Omega) \oplus L_\pi^2(\Omega),$$

where

$$L_\pi^2(\Omega) = \{\nabla p \in L^2(\Omega); p \in L_{\text{loc}}^2(\overline{\Omega})\}.$$

Let P be the projection (the Fujita-Kato projection) from $L^2(\Omega)$ onto $L_\sigma^2(\Omega)$ associated with the decomposition above. Then the Stokes operator $A : L_\sigma^2(\Omega) \rightarrow L_\sigma^2(\Omega)$ is defined by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega) \cap L_\sigma^2(\Omega), \quad Au = -P\Delta u.$$

In view of (NS.3), the linear operator $\mathcal{L} : L_\sigma^2(\Omega) \rightarrow L_\sigma^2(\Omega)$ we should consider is as follows:

$$\begin{cases} D(\mathcal{L}) = \{u \in D(A); (\omega \times x) \cdot \nabla u \in L^2(\Omega)\}, \\ \mathcal{L}u = Au - P[(\omega \times x) \cdot \nabla u - \omega \times u]. \end{cases}$$

It is proved that the operator \mathcal{L} is m -accretive, so that $-\mathcal{L}$ generates a (C_0) semigroup $\{e^{-t\mathcal{L}}; t \geq 0\}$ of contractions on $L^2_\sigma(\Omega)$. Furthermore, we have

$$(2) \quad \|u\|_{H^2(\Omega)} + \|P[(\omega \times x) \cdot \nabla u]\| \leq C \|(1 + \mathcal{L})u\|,$$

for all $u \in D(\mathcal{L})$ (see [8]). On account of unboundedness of the coefficient of \mathcal{L} , the elliptic regularity estimate (2) is no longer trivial. It is thus not so easy to show the closedness of \mathcal{L} directly. But the accretivity implies that \mathcal{L} is closable. So, we prove that $1 + \overline{\mathcal{L}}$ is surjective, where $\overline{\mathcal{L}}$ is the closure of \mathcal{L} . For the proof, we solve the corresponding stationary problem by using the solutions in \mathbb{R}^3 and in a bounded domain near the boundary Γ together with cut-off functions. For the recovery of the solenoidal condition in the localization, we make use of the result of Bogovskiĭ [1] on a continuous right-inverse of the divergence operator with zero boundary condition in bounded domains. At the next step, we show $\overline{\mathcal{L}} = \mathcal{L}$ together with estimate (2). The fractional powers of \mathcal{L} are also well defined as closed operators in $L^2_\sigma(\Omega)$, and we see that $D(\mathcal{L}^\alpha) \subset D(A^\alpha)$ with estimate

$$(3) \quad \|A^\alpha u\| \leq C_\alpha \|(1 + \mathcal{L})^\alpha u\|,$$

for all $u \in D(\mathcal{L}^\alpha)$ and $0 < \alpha \leq 1$. Indeed, (3) for the case $\alpha = 1$ is equivalent to (2), and the Heinz-Kato inequality for m -accretive operators (Tanabe [12, Chapter 2]) implies (3) for $0 < \alpha < 1$.

Our method to solve (NS.3) is to make use of the semigroup $e^{-t\mathcal{L}}$ together with the fractional powers of A and \mathcal{L} . Although this approach itself is, in principle, standard (see Fujita and Kato [4], Giga and Miyakawa [6]), the semigroup $e^{-t\mathcal{L}}$ is not

a usual one. The essential difficulty is the growth at space infinity of the coefficient $\omega \times x$ of the operator \mathcal{L} , so that the convection term $(\omega \times x) \cdot \nabla$ is not a perturbation of the Stokes operator A . In fact, the associated semigroup for the Cauchy problem in \mathbb{R}^3 is explicitly given by

$$(4) \quad [U(t)f](x) = O(t)^T [e^{t\Delta}f](O(t)x), \quad x \in \mathbb{R}^3, t > 0,$$

where

$$[e^{t\Delta}f](x) = (4\pi t)^{-3/2} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4t}} f(y) dy,$$

and it is proved that the semigroup $U(t)$ is never analytic on $L^2_\sigma(\mathbb{R}^3)$ (see [9]). This is a different feature caused by the convection term $(\omega \times x) \cdot \nabla$. Thus, we cannot expect that $e^{-t\mathcal{L}}$ is analytic. However, it has the remarkable smoothing effect. The following theorem asserts that $e^{-t\mathcal{L}}f$ is in $D(A)$ for all $t > 0$ whenever f is slightly smooth, and that $e^{-t\mathcal{L}}f$ is in $D(\mathcal{L})$ for all $t > 0$ under the additional assumption $(\omega \times x) \cdot \nabla f \in H^{-\infty}(\Omega) \equiv \bigcup_{s \geq 0} H^{-s}(\Omega)$.

Theorem 1. (i) Suppose that $f \in D(A^\delta)$ for some $0 < \delta \leq 1/2$. Then $e^{-t\mathcal{L}}f \in D(A)$ for all $t > 0$. Furthermore, there is a constant $C = C(\delta) > 0$ such that

$$(5) \quad \|Ae^{-t\mathcal{L}}f\| \leq C t^{-1+\delta} \|f\|_{D(A^\delta)},$$

for all $0 < t \leq 1$.

(ii) Suppose that $f \in D(A^\delta)$ for some $0 < \delta < 1$, and that $(\omega \times x) \cdot \nabla f \in H^{-s}(\Omega)$ for some $s \geq 0$. Then $e^{-t\mathcal{L}}f \in D(\mathcal{L})$ for all $t > 0$ and

$$\mathcal{L}e^{-t\mathcal{L}}f \in C(0, \infty; L^2_\sigma(\Omega)), \quad e^{-t\mathcal{L}}f \in C^1(0, \infty; L^2_\sigma(\Omega)),$$

with

$$\frac{d}{dt}e^{-t\mathcal{L}}f + \mathcal{L}e^{-t\mathcal{L}}f = 0, \quad t > 0,$$

in $L^2_\sigma(\Omega)$. Furthermore, there are constants $C = C(\delta) > 0$ and $C' = C'(s) > 0$ such that

$$(6) \quad \begin{aligned} \|\mathcal{L}e^{-t\mathcal{L}}f\| &\leq C (t \wedge 1)^{-1+\delta} \|f\|_{D(A^\delta)} \\ &\quad + C' (t \wedge 1)^{-s/2} \left\{ \|(\omega \times x) \cdot \nabla f\|_{H^{-s}(\Omega)} + \|f\| \right\}, \end{aligned}$$

for all $t > 0$, where $t \wedge 1 = \min\{t, 1\}$.

(iii) Let $0 < \delta < 1/2$. Then

$$\lim_{t \rightarrow 0} t^{1-\delta} \|Ae^{-t\mathcal{L}}f\| = 0,$$

for all $f \in D(A^\delta)$. For the same δ as above, let $0 \leq s < 2(1 - \delta)$. Then

$$\lim_{t \rightarrow 0} t^{1-\delta} \|\mathcal{L}e^{-t\mathcal{L}}f\| = 0,$$

for all $f \in D(A^\delta)$ satisfying $(\omega \times x) \cdot \nabla f \in H^{-s}(\Omega)$.

In Theorem 1 the case $\delta = 0$ (namely, $f \in L^2_\sigma(\Omega)$) is excluded on account of a technical difficulty caused by the solenoidal constraint. Indeed, in [7, Theorem 4] sharper results including $\delta = 0$ have been established for the realization of a model operator $\Delta + (\omega \times x) \cdot \nabla$ with the homogeneous Dirichlet boundary condition in $L^2(\Omega)$. But estimates (5) and (6) near $t = 0$ together with the fractional powers of A and \mathcal{L} are very useful for the proof of local existence of a unique solution to (NS.3). The strategy for the proof of Theorem 1 is as follows. We first derive the similar smoothing effect to Theorem 1 for the semigroup $U(t)$ given by (4). We next employ the method based on a refinement of the cut-off procedure developed in the proof of Theorem 4 of [7] combined with the result of Bogovskiĭ [1] mentioned above. For the details, see [9].

We now fix $\zeta \in C^\infty(\mathbb{R}^3)$ such that $0 \leq \zeta \leq 1$, $\zeta = 1$ near Γ and $\zeta = 0$ for large $|x|$, and put

$$(7) \quad b(x) = -\frac{1}{2} \nabla \times \{\zeta(x)|x|^2\omega\}.$$

Then $\nabla \cdot b = 0$ in Ω , $b = \omega \times x$ on Γ and $b = 0$ for large $|x|$. We set

$$\tilde{u}(x, t) = u(x, t) - b(x),$$

in (NS.3) and apply the projection P to the equation of motion to obtain the integral equation

$$(NS.4) \quad \tilde{u}(t) = e^{-t\mathcal{L}}[a - b] - \int_0^t e^{-(t-s)\mathcal{L}} P[\tilde{u} \cdot \nabla \tilde{u} + B\tilde{u}](s) ds, \quad t \geq 0,$$

in $L_\sigma^2(\Omega)$, where

$$B\tilde{u} = \tilde{u} \cdot \nabla b + b \cdot \nabla \tilde{u} + F[b],$$

$$F[b] = \Delta b + (\omega \times x) \cdot \nabla b - \omega \times b - b \cdot \nabla b.$$

The main theorem then reads as follows.

Theorem 2. Suppose that $a - b \in D(\mathcal{L}^\gamma)$ for some $1/4 < \gamma < 1/2$ and that $(\omega \times x) \cdot \nabla a \in H^{-s}(\Omega)$ for some $1 \leq s < 2(1 - \gamma)$. Then there exist $T > 0$ and a unique solution \tilde{u} to (NS.4) on the interval $[0, T]$, which is of class

$$\tilde{u} \in C([0, T]; L_\sigma^2(\Omega)),$$

and possesses the regularity $\tilde{u}(t) \in D(A)$, $0 < t \leq T$, with the properties:

$$(8) \quad \lim_{t \rightarrow 0} \|\tilde{u}(t) - (a - b)\|_{D(A^\gamma)} = \lim_{t \rightarrow 0} \|u(t) - a\|_{D(A^\gamma)} = 0,$$

$$(9) \quad \lim_{t \rightarrow 0} t^{\alpha-\gamma} \|\tilde{u}(t)\|_{D(A^\alpha)} = 0, \quad \gamma < \alpha \leq 1,$$

$$(10) \quad \|\tilde{u}(t)\|_{D(A^\alpha)} \leq C_\alpha K_0 t^{-\alpha+\gamma}, \quad 0 < t \leq T, \quad \gamma \leq \alpha \leq 1,$$

where

$$K_0 = \|a - b\|_{D(\mathcal{L}^\gamma)} + \|(\omega \times x) \cdot \nabla a\|_{H^{-s}(\Omega)} + \| |x|b \| + \|F[b]\|_{H^1(\Omega)}.$$

The proof is given in [9]. We conclude this article with some comments on Theorem 2.

Remark. (i) In view of (7), the assumption $a - b \in D(\mathcal{L}^\gamma) \subset D(A^\gamma)$ (see (3)) with $\gamma > 1/4$ implies that $a = \omega \times x$ on Γ (cf. Fujiwara [5]).

(ii) The critical case $\gamma = 1/4$ is the well known exponent of Fujita and Kato [4]. If Theorem 1 for $\delta = 0$ were deduced, then we could show Theorem 2 for the case $\gamma = 1/4$.

(iii) Under the assumption $(\omega \times x) \cdot \nabla a \in H^{-2(1-\gamma)}(\Omega)$, it is also possible to construct a unique solution. But the behavior (9) of such a solution is not clear.

(iv) The solution obtained in Theorem 2 is the so-called mild solution. Since we find the solution $\tilde{u}(t)$ with values in $D(A)$ and it does not belong to $D(\mathcal{L})$ in general, it seems to be difficult to derive the differentiability of \tilde{u} with respect to time t .

(v) Theorem 2 holds true with $\omega = (0, 0, 1)^T$ replaced by $\omega = (0, 0, \omega_0)^T$ for every $\omega_0 \in \mathbb{R}$. The existence interval $T = T(|\omega_0|) > 0$ is then monotonically decreasing with respect to $|\omega_0|$.

(vi) When the obstacle \mathcal{O} is not rotating, that is $\omega = 0$, the problem (NS.3) possesses a unique local strong solution for $a \in L_\sigma^3(\Omega) \supset D(A^{1/4})$, where $L_\sigma^3(\Omega)$ denotes the completion of $C_{0,\sigma}^\infty(\Omega)$ in $L^3(\Omega)$. If $\|a\|_{L^3(\Omega)}$ is sufficiently small, then the solution is extended globally in time. This is the result of Iwashita [10].

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